



CONCEPT DEVELOPMENT

Mathematics Assessment Project
CLASSROOM CHALLENGES
A Formative Assessment Lesson

Solving Linear Equations in Two Variables

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley
Beta Version

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Solving Linear Equations in Two Variables

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to formulate and solve problems using algebra and, in particular, to identify and help students who have the following difficulties:

- Solving a problem using two linear equations with two variables.
- Interpreting the meaning of algebraic expressions.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

A-CED: Create equations that describe numbers or relationships.

A-REI: Solve systems of equations.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*:

2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.

INTRODUCTION

This lesson is structured in the following way:

- Before the lesson, students work individually on the assessment task *Notebooks and Pens*. You then review their work and create questions for students to answer in order to improve their solutions.
- During the lesson, students work individually on a task that requires them to interpret and solve two equations in two variables. Students then compare and discuss their solutions in small groups.
- In the same small groups, students evaluate some sample solutions of the same task.
- In a whole-class discussion, students explain and compare the alternative solution strategies they have seen and used.
- Finally, students use what they have learned to revise their work on *Notebooks and Pens*.

MATERIALS REQUIRED

- Each individual student will need two copies of the assessment task *Notebooks and Pens* and a copy of the lesson task *Cash Registers*.
- Each small group of students will need a blank sheet of paper and copies of the four sheets *Sample Student Work*.
- Graph paper should be kept in reserve and used only when requested.
- Projector resources are provided to support the whole-class discussion.

TIME NEEDED

15 minutes before the lesson, a 1-hour lesson, and 10 minutes in a follow-up lesson (or for homework). Timings given are only approximate. Exact timings will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *Notebooks and Pens* (15 minutes)

Have the students do this task in class or for homework a day or more before the formative assessment lesson. This will give you an opportunity to assess the work and find out the kinds of difficulties students have with it. You will then be able to target your help more effectively in the follow-up lesson.

Give each student a copy of *Notebooks and Pens*.

Introduce the task briefly and help the class to understand the problem and its context.

Read through the questions, and try to answer them as carefully as you can.

Show all your work, so that I can understand your reasoning.

It is important that students are allowed to answer the questions without assistance, as far as possible.

Students should not worry too much if they cannot understand or do everything because there will be a lesson using a similar task, which should help them. Explain to students that by the end of the next lesson they should expect to answer questions such as these confidently. This is their goal.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding. The purpose of doing this is to forewarn you of issues that will arise during the lesson itself, so that you may prepare carefully.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it encourages students to compare their scores and distracts their attention from what they can do to improve their mathematics.


Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given on the next page. These have been drawn from common difficulties observed in trials of this lesson unit.

We suggest that you write a list of your own questions, based on your students' work, using the ideas below. You may choose to write questions on each student's work. If you do not have time to do this, select a few questions that will be of help to the majority of students. These can be written on the board at the end of the lesson.

Notebooks and Pens


A store sells pens at \$2 and notebooks at \$5.
 n = number of notebooks sold.
 p = number of pens sold.
The following equations are true:
 $4n = p$
 $5n + 2p = 39$

Here is what Dan and Emma think the equations mean:



Dan

I think the first equation means that the store sells four times as many notebooks as pens.



Emma

I think the second equation means that the store sold 5 notebooks and 2 pens.

Is Dan correct?
If you think Dan is wrong, explain the mistake and explain what you think $4n = p$ means.

Is Emma correct?
If you think Emma is wrong, explain the mistake and explain what you think $5n + 2p = 39$ means.

Figure out for yourself the number of pens and the number of notebooks sold in the store.
Number of pens sold = Number of notebooks sold =

Common issues:**Suggested questions and prompts:**

Student assumes that the letter stands for an object not a number For example: The student says that the statements are correct. Or: The student realizes the equations are incorrect, but is unable to explain why.	<ul style="list-style-type: none">• What does the letter p represent?• Write the equation as a sentence. Does your sentence match what Dan/Emma said?• If $n = 3$, what would p equal in the first equation? Which is greater: n or p?• Are there more notebooks than pens? How do you know?
Student only uses one equation For example: The student finds a value or values for n and p that fits one equation but not the other, such as $n = 1$ and $p = 4$ for the first equation.	<ul style="list-style-type: none">• For this equation, is there another pair of values for n and p? And another? How do you know which value is correct?• How can you check that your values for n and p work for both equations?
Student produces unsystematic guess and check work For example: The student works out three or four seemingly unconnected combinations of values for n and p .	<ul style="list-style-type: none">• What is a sensible value to try for n (or p)? Why?• Can you organize your work in a table?
Student provides poor explanation For example: The student presents the work as a series of unexplained numbers and/or calculations.	<ul style="list-style-type: none">• Would someone unfamiliar with your type of solution easily understand your work?• Have you explained how you arrived at your answer?
Student makes algebraic mistakes For example: The student makes a mistake when manipulating the algebra in the equations.	<ul style="list-style-type: none">• How can you check that your answer is correct?
Student solves the two equations correctly Student needs an extension task.	<ul style="list-style-type: none">• Can you now use a different method, for example, a table, a graph, or algebra?• Is this method better than your original one? Why?

SUGGESTED LESSON OUTLINE

Individual work: *Cash Registers* (10 minutes)

Give each student the task sheet *Cash Registers*. Help students to understand the problem, and explain the context of the task briefly.

Spend 10 minutes on your own answering these questions.

What does ‘simultaneously’ mean?

Show all your work on the sheet.

Students who sit together often produce similar answers and, when they come to compare their work, they have little to discuss.

For this reason we suggest that, when students do this task individually, you ask them to move to different seats. Then, for the collaborative task, allow them to return to their usual places. Experience has shown that this produces more profitable discussions.

Collaborative small-group work: *Cash Registers* (10 minutes)

Organize the class into small groups of two or three students and hand out a fresh sheet of paper to each group. Students should now have another go at the task, but this time they will combine their ideas.

I want you now to work together in your groups.

Your task is to produce an answer together that is better than your individual ones.

Throughout this activity, encourage students to articulate their reasoning, justify their choices mathematically, and question the choices put forward by others.

As students work you have two tasks: to note student approaches to their work and to support their thinking.

Note student approaches to their work

How do students choose to tackle this task? Notice the variety in approaches. Notice any common errors. You can use this information to focus your questioning in the whole-class discussion towards the end of the lesson.

Support student thinking

Try not to make suggestions that prompt students towards a particular answer. Instead, ask questions to help students clarify their thinking.

You may find that some students interpret the letters as ‘quarters’ and ‘dollars’ rather than the number of quarters and number of dollars. For example, they may say things like:

“ $3x = y$ means three times as many quarters as dollars.”

“ $4x + y = 70$ means 4 quarters plus dollars equals 70.”

“There is \$70 in the till.”

The following questions and prompts may be helpful for both students struggling with the task and those making quick progress:

Cash Registers


The drawer of a cash register contains some quarters and some dollar bills.

x = the number of quarter coins in the cash register.
 y = the number of dollar bills in the cash register.

The following two equations are true:

$$3x = y$$
$$4x + y = 70$$

1. Explain in words the meaning of each equation.
2. Find two pairs of values for x and y that satisfy the first equation.
3. Find two pairs of values for x and y that satisfy the second equation.
4. Find pairs of values for x and y that satisfy both equations *simultaneously*.



What do the letters x and y represent?

Replace x and y in this equation by words and now say what the equation means.

Are there more dollar bills or more quarters in the cash register? How do you know?

Do you have any values for x and y that work for the first equation? How can you check to see if they also work for the second one? If these don't fit, what other values for x and y can you use?

Why have you chosen these values for x and y ?

Suppose there are 5 quarters in the drawers of the cash register, so $x = 5$. From the first equation, how many dollar bills are there? [15.] From the second equation, how many dollar bills are there? [50.] There cannot be both 15 and 50 dollar bills!

Can you find a value for x that will give the same answer in both cases?

How can you check that your answer is right?

Can you use these equations to calculate the amount of money in the cash register?

If the whole class is struggling on the same issue, you may want to write a couple of questions on the board and organize a brief whole-class discussion. You could also ask students who performed well in the assessment to help struggling students.

Collaborative analysis of Sample Student Work (15 minutes)

When all groups have made a reasonable attempt, ask them to put their work to one side. Give each group copies of the *Sample Student Work*. This task will give students the opportunity to discuss and evaluate possible approaches to the task without providing a complete solution strategy.

Ideally, all groups will review all four pieces of work. However, if you are running out of time, choose just two solutions for all groups to analyze, using what you have learned during the lesson about what students find most difficult.

Encourage students to think more deeply using the following questions. (These are reproduced on the projector resource *Assessing Sample Student Work*.)

You are the teacher and have to assess this work.

Correct the work and write comments on the accuracy and organization of each response.

- *What do you like about the work?*
- *What method did the student use? Is it clear? Is it accurate? Is it efficient?*
- *What errors did the student make?*
- *How might the work be improved?*

During this small-group work, support the students as before. Also, check to see which of the explanations students find more difficult to understand. Note similarities and differences between the sample approaches and those the students used in the group work.

Whole-class discussion: comparing different approaches (15 minutes)

Hold a whole-class discussion to consider the different approaches used in the sample work. Focus the discussion on those parts of the task that the students found difficult. Ask representatives from each group to explain and critique one student's method from the *Sample Student Work*. During the discussion you may find it helpful to use the projector resources, which show the different sample solutions.

Which approach did you like best? Why?

Which approach did you find most difficult to understand? Why?

Next lesson: Improve individual responses to *Notebooks and Pens* (10 minutes)

Have students do this task at the beginning of the next lesson if you do not have time during the lesson itself. Some teachers like to set this task for homework.

Return the students' individual work on the assessment task *Notebooks and Pens* along with a second blank copy of the task sheet.

Look at your original responses and think about what you have learned this lesson.

Using what you have learned, try to improve your work.

If you have not written questions on individual pieces of work then write your list of questions on the board. Students are to select from this list only those questions appropriate to their own work.

SOLUTIONS

Assessment task: *Notebooks and Pens*

I think the first equation means that the store sells four times as many notebooks as pens.



Dan



Emma

I think the second equation means that the store sold 5 notebooks and 2 pens.

Dan is incorrect:

Dan has misinterpreted n to mean, 'notebooks sold' rather than '**the number of** notebooks sold.'

So he has read the equation ' $4n = p$ ' as "there are four notebooks sold for every single pen sold."

The equation actually means, '4 times the number of notebooks sold equals the number of pens sold,' or 'the store sells four times more pens than notebooks.'

Emma is incorrect:

Emma has also misinterpreted n to mean 'notebooks' rather than '**the number of** notebooks.'

In the second statement, $5n$ does not mean, 'there are 5 notebooks.' It means '5 times the number of notebooks.'

Since each notebook costs \$5, $5n$ gives you the amount of money taken from selling notebooks, and since each pen costs \$2, $2p$ gives you the amount of money taken by selling pens. So $5n + 2p = 39$ means that \$39 was taken altogether by selling notebooks and pens at these prices. However, the equation does not, in isolation, tell you how many notebooks or pens were sold.

Using the first equation to substitute $4n$ for p in the second equation gives $n = 3$ and $p = 12$.

3 notebooks and 12 pens were sold.

Cash Registers

- The number of dollar bills is three times the number of quarters.
Four times the number of quarters plus the number of dollar bills totals 70.
- Possible values: (4, 12) or (8, 24).
- Possible values: (12, 22) or (7, 42).
- $x = 10, y = 30$

Comments on Sample Student Work

The image shows two sections of handwritten student work. The top section, labeled 'Ava', shows a 'guess and check' method. It starts with the equations $3x = y$ and $4x + y = 70$. The student lists several pairs of values for x and y and checks them in the second equation. The pairs are (1, 3), (2, 6), (3, 9), and (4, 12). The checks are $4(1) + 3 = 7$ (crossed out), $4(2) + 6 = 14$ (crossed out), $4(3) + 9 = 21$ (crossed out), and $4(4) + 12 = 28$ (crossed out). The bottom section, labeled 'Ethan', shows an elimination method. It starts with the same equations. The student subtracts the first equation from the second to get $7x + 2y = 70$. Then, they substitute $x = 5$ to get $7(5) + 2y = 70$, which simplifies to $2y = 35$. They then divide by 2 to get $y = 17.5$, noting that it 'must be a whole number'. Finally, they substitute $x = 6$ into the first equation to get $4(6) + 2y = 70$, which simplifies to $2y = 28$, and then $y = 14$.

Ava used 'guess and check' with both equations

Strengths: Her work is systematic and easy to follow.

Weaknesses: Her method is inefficient and, although it is systematic, she has not reflected on each answer to determine the next set of values to check.

Her lack of progress leads to her abandoning the task.

Ava could add an explanation about her solution method.

Ethan used an elimination method

Strengths: This method can work if equations are manipulated carefully.

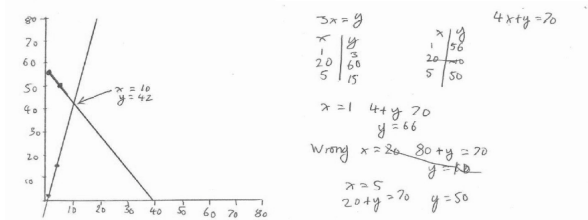
Weaknesses: Ethan makes a mistake when rearranging the first equation. Consequently, when the two equations are added together, a variable is not eliminated, but instead Ethan has created an equation with two variables.

Ethan briefly used 'guess and check.' This gives many solutions. Ethan has simply opted to figure out two solutions. Both answers are incorrect. Ethan has not explained his working or why he was happy with the second set of values.

If the first equation had been $3x + y = 0$, what would still be wrong with Ethan's method?

Would this method ever obtain just one solution?

$$\begin{aligned}
 3x &= y & x &= \frac{y}{3} \\
 4x + y &= 70 \\
 4 \times \frac{y}{3} + y &= 70 \\
 \times 3 \quad 4y + y &= 210 \\
 5y &= 210 \\
 y &= 42 \\
 x &= \frac{42}{3} = 14
 \end{aligned}$$



Joe used a substitution method

Strengths: This is an efficient method.

Weaknesses: Joe failed to multiply all the terms on the left-hand side of the equation by three, so he obtained an incorrect answer.

If Joe had substituted $3x$ for y into the second equation the solution would have been very straightforward.

Mia used a graphical approach

Strengths: This method can work.

Weaknesses: In this case a graphical approach is not a very efficient strategy.

Mia has made an error in her second table: $y = 66$ not 56 .

Mia could have used the co-ordinates $(20, -10)$ to help plot the second line. There are no labels on either axis. The scale of Mia's graph means that the lines are not plotted accurately.

Was Mia right to abandon $(20, -10)$ as a point to be used to plot the second line?

Notebooks and Pens

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p = number of pens sold.

The following equations are true:

$$4n = p$$

$$5n + 2p = 39$$



Here is what Dan and Emma think the equations mean:

I think the first equation means that the store sells four times as many notebooks as pens.



Dan



Emma

I think the second equation means that the store sold 5 notebooks and 2 pens.

Are Dan and Emma correct?

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If you think Dan is wrong, explain the mistake and explain what *you* think the equation means.

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If you think Emma is wrong, explain the mistake and explain what *you* think the equation means.

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Figure out for yourself the number of pens and the number of notebooks sold in the store.

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Cash Registers

The drawer of a cash register contains some quarters and some dollar bills.

x = the number of quarter coins in the cash register.

y = the number of dollar bills in the cash register.

The following two equations are true:

$$3x = y$$

$$4x + y = 70$$



1. Explain in words the meaning of each equation.

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2. Find two pairs of values for x and y that satisfy the first equation.

3. Find two pairs of values for x and y that satisfy the second equation.

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4. Find pairs of values for x and y that satisfy both equations *simultaneously*.

This image shows a single sheet of white paper with horizontal blue or grey ruling lines. The lines are evenly spaced and run across the width of the page. There are no margins, text, or other markings on the paper.

Sample Student Work: Ava

$$\begin{array}{ll} 3 & x=y \quad y=3x \\ \text{Try } x=1 & y=3 \\ \text{Try } x=2 & y=6 \\ \text{Try } x=3 & y=9 \\ \text{Try } x=4 & y=12 \end{array} \quad \begin{array}{l} 4x+y=70 \\ 4x+y=7 \quad \times \\ 8+6=14 \quad \times \\ 12+9=21 \quad \times \\ 16+12=28 \quad \times \end{array}$$

You are the teacher and have to assess this work.

- What do you like about Ava's work?
- What method did she use? Is it clear? Is it accurate? Is it efficient?
- What errors did Ava make?
- How might her work be improved?

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Sample Student Work: Ethan

$$\begin{array}{l} 3x = y \\ 4x + y = 70 \\ 3x + y = 0 \\ 4x + y = 70 + \\ \hline 7x + 2y = 70 \\ x = 53 \quad 7 \times 5 = 35 \\ 35 + 2y = 70 \\ 2y = 35 \\ y = 17.5 - \text{must be a whole} \\ \text{number} \\ \textcircled{x = 6} \quad 2 \times 6 = 42 \\ 42 + 2y = 70 \\ 2y = 28 \\ y = 14 \end{array}$$

You are the teacher and have to assess this work.

- What do you like about Ethan's work?
- What method did he use? Is it clear? Is it accurate? Is it efficient?
- What errors did Ethan make?
- How might his work be improved?

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Sample Student Work: Joe

$$\begin{aligned} 3x &= y & x &= \frac{y}{3} \\ 4x + y &= 70 \\ 4 \times \frac{y}{3} + y &= 70 \\ \times 3 \quad 4y + y &= 210 \\ 5y &= 210 \\ y &= 42 \\ x &= \frac{42}{3} = 14 \end{aligned}$$

You are the teacher and have to assess this work.

- What do you like about Joe's work?
- What method did Joe use? Is it clear? Is it accurate? Is it efficient?
- What errors did Joe make?
- How might the work be improved?

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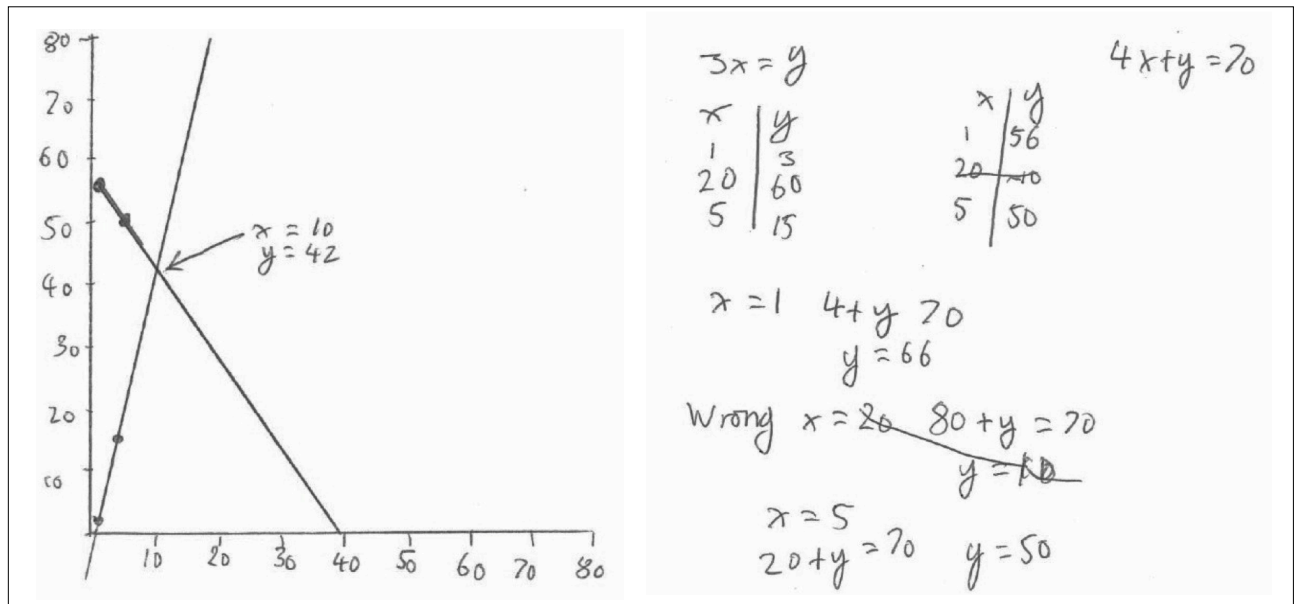
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Sample Student Work: Mia



You are the teacher and have to assess this work.

- What do you like about Mia's work?
- What method did she use? Is it clear? Is it accurate? Is it efficient?
- What errors did Mia make?
- How might her work be improved?

Assessing Sample Student Work

You are the teacher and have to assess this work.

Correct the work and write comments on the accuracy and organization of each response.

- What do you like about this student's work?
- What method did the student use?
Is it clear? Is it accurate? Is it efficient?
- What errors did the student make?
- How might the work be improved?

Sample Student Work: Ava

$$3 \quad x = y \quad y = 3x$$

$$4x + y = 70$$
$$4\cancel{x} + y = 7 \quad \times$$

$$\text{Try } x = 1 \quad y = 3$$

$$\text{Try } x = 2 \quad y = 6$$

$$8 + 6 = 14 \quad \times$$

$$\text{Try } x = 3 \quad y = 9$$

$$12 + 9 = 21 \quad \times$$

$$\text{Try } x = 4 \quad y = 12$$

$$16 + 12 = 28 \quad \times$$

Sample Student Work: Ethan

$$\begin{aligned} 3x &= y \\ 4x + y &= 70 \\ 3x + y &= 0 \\ 4x + y &= 70 + \\ \hline 7x + 2y &= 70 \end{aligned}$$

$x = 53 \quad 7 \times 5 = 35$

$$\begin{aligned} 35 + 2y &= 70 \\ 2y &= 35 \\ y &= 17.5 - \text{must be a whole number} \end{aligned}$$

$x = 6$

$$\begin{aligned} 7 \times 6 &= 42 \\ 42 + 2y &= 70 \\ 2y &= 28 \\ y &= 14 \end{aligned}$$

Sample Student Work: Joe

$$3x = y$$

$$x = \frac{y}{3}$$

$$4x + y = 70$$

$$4 \times \frac{y}{3} + y = 70$$

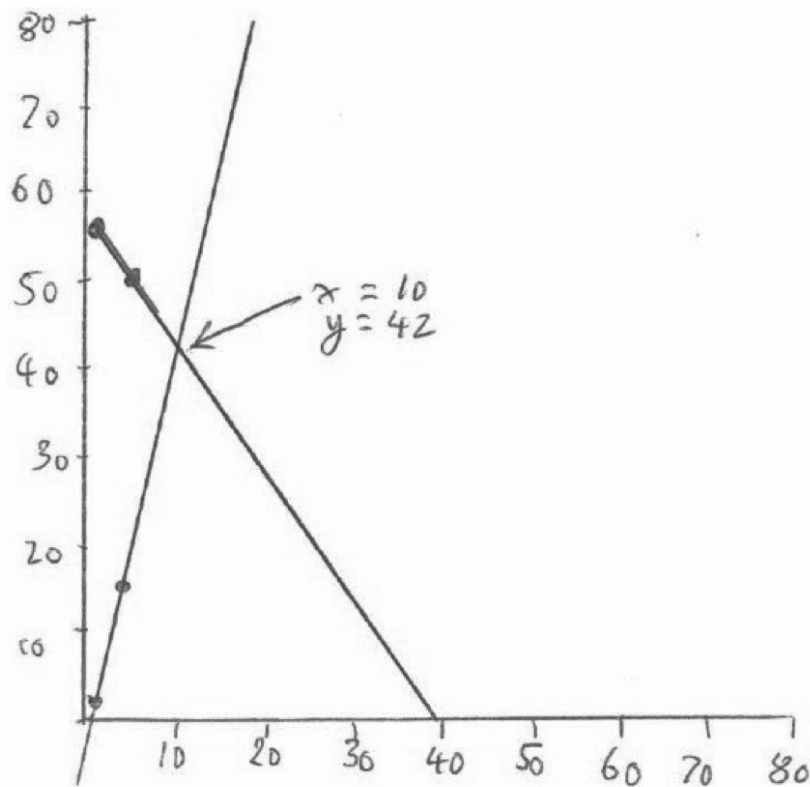
$$\times 3 \quad 4y + y = 210$$

$$5y = 210$$

$$y = 42$$

$$x = \frac{42}{3} = 14$$

Sample Student Work: Mia



$$3x = y$$

x	y
1	3
20	60
5	15

$$4x + y = 70$$

x	y
1	56
20	40
5	50

$$x = 1 \quad 4 + y = 70$$

$$y = 66$$

Wrong $x = 20 \quad 80 + y = 70$

$$y = -10$$

$$x = 5 \quad 20 + y = 70$$

$$y = 50$$

Mathematics Assessment Project

CLASSROOM CHALLENGES

This lesson was designed and developed by the
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It was refined on the basis of reports from teams of observers led by
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based on their observation of trials in US classrooms
along with comments from teachers and other users.

This project was conceived and directed for
MARS: Mathematics Assessment Resource Service
by
Alan Schoenfeld, Hugh Burkhardt, Daniel Pead, and Malcolm Swan
and based at the University of California, Berkeley

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